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On the Σ-invariants of Artin groups

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Abstract

We use the action of an Artin group on its associated Deligne complex (as defined by Charney and Davis) to give information about the Σ-invariants of Artin groups. This gives a nearly complete description of the finiteness properties of normal subgroups above the commutator subgroup of an Artin group. © 2001 Elsevier Science B.V. All rights reserved.

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Introduction

The study of Artin groups has received renewed enthusiasm. This is partially because Artin groups have relatively simple and combinatorial presentations which allow one to exhibit pathological behaviours. (See, for example, [3,14].) Further, while Artin groups have been studied for some time, there are very few complete results which hold for all Artin groups. In particular, the word and $K(\pi, 1)$ problems have been solved only in certain cases. (See [1,2,11].)

Given a finite simplicial graph $\mathcal{G}$, with edges labeled by integers greater than one, the associated Artin group, denoted $A_{\mathcal{G}}$, has a finite presentation with generators corresponding to the vertices of $\mathcal{G}$, and relations

$$\underbrace{vwv\cdots}_{n \text{ letters}} = \underbrace{wvw\cdots}_{n \text{ letters}},$$

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where $\{v, w\}$ is an edge of $\mathcal{G}$ labeled n. Standard examples of Artin groups are the braid groups and the fundamental groups of $(2, n)$-torus link complements.

Given any Artin group $A_{\mathcal{G}}$ there is an associated Coxeter group $C_{\mathcal{G}}$ which is the quotient of $A_{\mathcal{G}}$ formed by adding the relations $v^2 = 1$ for each generator v. An Artin group is of *finite type* if its associated Coxeter group is finite. Braid groups are Artin groups of finite type; their associated Coxeter groups are the symmetric groups. The fundamental groups of $(2, n)$-torus link complements are also Artin groups of finite type. They correspond to Artin groups where $\mathcal{G}$ is a single edge labeled n and their associated Coxeter groups are the finite dihedral groups.

Given an Artin group $A_{\mathcal{G}}$, let $\widehat{\mathcal{G}}$ be the simplicial complex formed by attaching a simplex σ of appropriate dimension to each complete subgraph $\mathcal{C} \subset \mathcal{G}$ for which $A_{\mathcal{C}}$ is an Artin group of finite type. Notice that $\mathcal{G} \hookrightarrow \widehat{\mathcal{G}}$ because Artin groups $A_{\mathcal{G}}$—where $\widehat{\mathcal{G}}$ is a single edge—are of finite type. In this paper we give a partial description of the Σ-invariants of Artin groups in terms of the topology of $\widehat{\mathcal{G}}$. This is very much in the spirit of [17], where the Σ-invariants of right-angled Artin groups (also known as graph groups) were completely determined. While we use similar techniques, we are unable to get complete answers in the general case.

The Σ-invariants of a group G (which for our purposes we assume is of type $\mathcal{F}_\infty$) are geometric subsets of a sphere. More specifically, the complement of the zero map in the real vector space $\mathrm{Hom}(\mathrm{G}, \mathbb{R})$ is called the set of *characters* of a group G. For any character χ, let $[\chi] = \{r\chi \mid 0 < r \in \mathbb{R}\}$ be a ray in $\mathrm{Hom}(\mathrm{G}, \mathbb{R})$; the set of all such rays is denoted $S(G)$ and can be realized as a 'sphere' inside $\mathrm{Hom}(\mathrm{G}, \mathbb{R})$. Each of the Σ-invariants of G is a subset of $S(G)$. Among other things, these invariants determine the finiteness properties of all the normal subgroups above the commutator subgroup of G.

There are two standard sequences of Σ-invariants: the homotopical invariants $S(G) \supseteq \Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots$, and the homological invariants $S(G) \supseteq \Sigma^1(G, \mathbb{Z}) \supseteq \Sigma^2(G, \mathbb{Z}) \supseteq \cdots$. Their complements in $S(G)$ are usually denoted by $\Sigma^n(G)^c$ and $\Sigma^n(G, \mathbb{Z})^c$. The homotopical invariants determine the $\mathcal{F}_m$ properties of normal subgroups above the commutator, while the homological invariants determine their FP_m properties. The first invariants in these sequences, $\Sigma^1(G)$ and $\Sigma^1(G, \mathbb{Z})$, are the same and were introduced in [5]. The higher invariants were introduced in [6,23]. While $\Sigma^n(G) \subseteq \Sigma^n(G, \mathbb{Z})$ for any G and n, the work of Bestvina and Brady shows they are not always equal [3]; examples are given in [17] for which $\Sigma^n(G)$ is empty while $\Sigma^n(G, \mathbb{Z})$ is dense in $S(G)$. Both sequences of invariants correspond to certain 'connectivity' properties of certain 'half spaces' in a classifying space for G; formal definitions are given in Section 2.

While some progress towards computation of Σ^1 for Artin groups was made by the first author in [16] not much is known about the higher invariants. In this introduction we therefore concentrate on the higher invariants, postponing discussion of $\Sigma^1(A_{\mathcal{G}})$ until Section 3.

A key fact we will use in this paper is that every Artin group of finite type contains an element Δ where $\langle \Delta \rangle = \mathbb{Z}$ and Δ (or at worst Δ^2) is central [10]. If σ is a simplex in $\widehat{\mathcal{G}}$, we let Δ_σ denote this element in the corresponding subgroup of finite type $A_{\mathcal{C}} < A_{\mathcal{G}}$.

Theorem A. *Let $A_{\mathcal{G}}$ be an Artin group with associated simplicial complex $\widehat{\mathcal{G}}$, and let $\chi : A_{\mathcal{G}} \to \mathbb{R}$. If $\widehat{\mathcal{G}}$ is 1-connected (respectively 1-acyclic) and for any subsimplex $\sigma \subset \widehat{\mathcal{G}}$, $\chi(\Delta_\sigma) \neq 0$, then $[\chi] \in \Sigma^2(A_{\mathcal{G}})$ (respectively $[\chi] \in \Sigma^2(A_{\mathcal{G}}, \mathbb{Z})$).*

Corollary A.1. *Under the hypotheses of Theorem A, if $\chi(A_{\mathcal{G}})$ is infinite cyclic, then* $\ker \chi$ *is finitely presented (respectively* FP_2*).*

Corollary A.2. *Let $A_{\mathcal{G}}$ be an Artin group with 1-connected (respectively 1-acyclic) associated simplicial complex $\widehat{\mathcal{G}}$. Then $\Sigma^2(A_{\mathcal{G}})^c$ (respectively $\Sigma^2(G, \mathbb{Z})^c$) is contained in the intersection of the character sphere $S(A_{\mathcal{G}})$ with the union of finitely many hyperplanes in* $\mathrm{Hom}(\mathrm{A}_{\mathcal{G}}, \mathbb{R})$. *Hence $\Sigma^2(A_{\mathcal{G}})$ (respectively $\Sigma^2(G, \mathbb{Z})$) is dense in $S(A_{\mathcal{G}})$.*

An Artin group is of *FC type* if all the subgroups corresponding to complete subgraphs of $\mathcal{G}$ are of finite type. In this case, each complete subgraph $\mathcal{C} \subset \mathcal{G}$ spans a simplex σ in $\widehat{\mathcal{G}}$. The Artin group $A_{\mathcal{G}}$ is of FC type if and only if $\widehat{\mathcal{G}}$ is a *F*lag *C*omplex, whence the "FC" terminology. (Right-angled Artin groups are Artin groups of FC type.) Charney and Davis have described CAT(0) complexes on which Artin groups of FC type act, but the action is not proper [11]. However, combining known facts about Artin groups with this action and techniques from [17], we are able to establish the following results.

Theorem B. *Let $A_{\mathcal{G}}$ be an Artin group of FC type with associated simplicial complex $\widehat{\mathcal{G}}$, and let $\chi : A_{\mathcal{G}} \to \mathbb{R}$. If $\widehat{\mathcal{G}}$ is $(n-1)$-connected (respectively $(n-1)$-acyclic) and for any subsimplex $\sigma \subset \widehat{\mathcal{G}}$, $\chi(\Delta_\sigma) \neq 0$, then $[\chi] \in \Sigma^n(A_{\mathcal{G}})$ (respectively $[\chi] \in \Sigma^n(A_{\mathcal{G}}, \mathbb{Z})$).*

Corollary B.1. *Under the hypotheses of Theorem B, if $\chi(A_{\mathcal{G}})$ is infinite cyclic, then* $\ker \chi$ *is $\mathcal{F}_n$ (respectively* FP_n*).*

Corollary B.2. *Let $A_{\mathcal{G}}$ be an Artin group of FC type where $\widehat{\mathcal{G}}$ is $(n-1)$-connected (respectively $(n-1)$-acyclic). Then $\Sigma^n(A_{\mathcal{G}})^c$ (respectively $\Sigma^n(A_{\mathcal{G}}, \mathbb{Z})^c$) is contained in the intersection of the character sphere $S(A_{\mathcal{G}})$ with the union of finitely many hyperplanes in* $\mathrm{Hom}(\mathrm{A}_{\mathcal{G}}, \mathbb{R})$. *Hence $\Sigma^n(A_{\mathcal{G}})$ (respectively $\Sigma^n(A_{\mathcal{G}}, \mathbb{Z})$) is dense in $S(A_{\mathcal{G}})$.*

Our paper is organized as follows. In Section 1 we briefly outline the relevant background material on Artin groups. In Section 2 we do the same for the Σ-invariants. In Section 3 we use the description of Σ^1 in terms of actions on $\mathbb{R}$-trees to give a quick proof of the partial description of $\Sigma^1(A_{\mathcal{G}})$ from [16]. This is followed by proofs of our main theorems in Section 4. Finally, Section 5 contains the complete description of the Σ-invariants in the most simple case, namely where the graph $\mathcal{G}$ is a tree.

1. Background on Artin groups

Let $A_{\mathcal{G}}$ be an Artin group with defining graph $\mathcal{G}$. Charney and Davis constructed a cubical complex, the modified Deligne complex, on which $A_{\mathcal{G}}$ acts [11]. Let $\mathcal{S}^f$ be the poset of complete subgraphs of $\mathcal{G}$ which correspond to Artin groups of finite type, ordered by inclusion. The 'empty subgraph' is included in $\mathcal{S}^f$, and hence the geometric realization of $\mathcal{S}^f$ is the cone over the barycentric subdivision of $\widehat{\mathcal{G}}$. In particular, one can give $\mathcal{S}^f$ a cubical structure where the induced structure on the boundary of $\mathcal{S}^f$ is equal to the cubulation of $\widehat{\mathcal{G}}$.

Let

$$\mathcal{DS}^f = \{aA_{\mathcal{C}} \mid a \in A_{\mathcal{G}}, \mathcal{C} \in \mathcal{S}^f\}$$

be the geometric realization of this (much larger) poset ordered by inclusion. Clearly $A_{\mathcal{G}}$ acts on $\mathcal{DS}^f$ on the left with a copy of $\mathcal{S}^f$ as fundamental domain. The stabilizers of the cells of $\mathcal{DS}^f$ under this action are the conjugates of the Artin groups of finite type corresponding to simplices $\sigma \subset \widehat{\mathcal{G}}$. (For example, the stabilizer of the cell $aA_{\mathcal{C}_1} \subset aA_{\mathcal{C}_2} \subset \cdots \subset aA_{\mathcal{C}_k}$ is $aA_{\mathcal{C}_1}a^{-1}$.) The stabilizers of the cells inject into the entire group $A_{\mathcal{G}}$ by [15], and hence by Haefliger's work on complexes of groups [13] the complex $\mathcal{DS}^f$ is 1-connected.

If the cubical complex $\mathcal{DS}^f$ is given a piecewise Euclidean metric such that each cube is isometric to the unit cube of the appropriate dimension, then $\mathcal{DS}^f$ is CAT(0) if and only if $A_{\mathcal{G}}$ is of FC type [11]. In this case we know $\mathcal{DS}^f$ is contractible, so we can achieve more general results.

The most important fact we will need is that the cell stabilizers are Artin groups of finite type. Every Artin group of finite type can be decomposed as the direct sum of *irreducible* Artin groups of finite type. These are the Artin groups whose corresponding Coxeter groups are irreducible; a complete list of these is given on pages 22–23 in [9]. Also, all Artin groups of finite type have finite $K(A_{\mathcal{G}}, 1)$'s. We briefly discuss what is known about the Σ-invariants of Artin groups of finite type in Section 2.

2. Background on the Σ-invariants

Let $\chi \in \mathrm{Hom}(G, \mathbb{R}) - \{0\}$, and let G_χ denote the monoid of elements $\{g \in G \mid \chi(g) \geqslant 0\}$. The map χ represents a point in $\Sigma^1(G)$ if the subgraph of the Cayley graph of G induced by G_χ is connected. The definitions for the higher invariants are quite a bit more technical. For convenience, let G be a group and let KG be a $K(G,1)$ that is finite in all dimensions, where $KG^{(0)}$ is a single vertex; denote the universal cover of KG by $\widetilde{KG}$. In this context, the Σ-invariants have fairly geometric descriptions. Corresponding to any character $\chi : G \to \mathbb{R}$ one can define a map $\widetilde{\chi} : \widetilde{KG} \to \mathbb{R}$; the map $\widetilde{\chi}$ is defined on the vertices of $\widetilde{KG}$ by $\widetilde{\chi}(v) = \chi(g)$ if $v = b \cdot g$ for some fixed base vertex b, and is extended linearly and G-equivariantly from the vertices to the entire universal cover. Let $\widetilde{KG}_\chi^{[a,\infty)}$ denote the maximal subcomplex in $\widetilde{KG} \cap \widetilde{\chi}^{-1}[a, \infty)$. For any non-negative constant d, the inclusion $\widetilde{KG}_\chi^{[0,\infty)} \hookrightarrow \widetilde{KG}_\chi^{[-d,\infty)}$ induces a map between reduced homology groups $\widetilde{H}_i(\widetilde{KG}_\chi^{[0,\infty)}, \mathbb{Z}) \to \widetilde{H}_i(\widetilde{KG}_\chi^{[-d,\infty)}, \mathbb{Z})$ and a map between homotopy groups $\pi_i(\widetilde{KG}_\chi^{[0,\infty)}) \to \pi_i(\widetilde{H}_i(\widetilde{KG}_\chi^{[-d,\infty)})$. A character χ represents a point in $\Sigma^n(G, \mathbb{Z})$ (respectively $\Sigma^n(G)$) if and only if there exists a non-negative constant d such that the induced map on the reduced homology groups (respectively homotopy groups) is zero for $i < n$. This definition is independent of choice of $K(G,1)$ and $\widetilde{\chi}$.

Let G be as above, let $0 \neq \chi : G \to \mathbb{R}$, and let $h : \widetilde{KG} \to \mathbb{R}$ be a (G,χ)-equivariant height function, that is, $h(g \cdot x) = \chi(g) + h(x)$ for all $g \in G$ and $x \in \widetilde{KG}$. Then:

Σ^n-criterion [23,7]. *$[\chi] \in \Sigma^n(G)$ if and only if there exists a continuous, cellular, G-equivariant map $\phi : \widetilde{KG}^{(n)} \to \widetilde{KG}^{(n)}$ such that $h(\phi(x)) > h(x)$ for all $x \in \widetilde{KG}^{(n)}$.*

The Σ-invariants of G determine the finiteness properties of subgroups above the commutator subgroup of G. Recall that a group is $\mathcal{F}_m$ if there is a $K(G,1)$ with finite m-skeleton, while a group is FP_m if $\mathbb{Z}$ has a projective resolution as a trivial $\mathbb{Z}G$-module which is finite through dimension m. It is easy to show that $\mathcal{F}_m \Rightarrow \mathrm{FP}_m$ but only recently has it been established that the FP_m condition is strictly weaker in dimensions greater than one [3]. Every map $\phi : G \twoheadrightarrow \mathbb{Z}^n$ determines a corresponding $(n-1)$-sphere in $S(G)$, and the kernel of this map is $\mathcal{F}_m$ (respectively FP_m) if and only if this sphere is contained in $\Sigma^m(G)$ (respectively $\Sigma^m(G,\mathbb{Z})$). At the lowest level, the kernel of a map $\phi : G \twoheadrightarrow \mathbb{Z}$ is $\mathcal{F}_m$ or FP_m if and only if $[\phi]$ and $[-\phi]$ are both contained in $\Sigma^m(G)$ or $\Sigma^m(G,\mathbb{Z})$, respectively (see [5,6]).

If an $\mathcal{F}_n$ group G contains central elements, the following lemma provides a sufficient but certainly not necessary condition for a character to represent a point in $\Sigma^n(G)$.

Lemma 2.1. *Let G be a group of type $\mathcal{F}_n$, let $\chi : G \to \mathbb{R}$ be a character, and let $c \in G$ be central. If $\chi(c) \neq 0$, then $[\chi] \in \Sigma^n(G) \subseteq \Sigma^n(G,\mathbb{Z})$.*

Proof. We may assume that $\chi(c) > 0$; if $\chi(c) < 0$ then replace "c" in this proof by "c^{-1}". Since c is central, the map $\phi : \widetilde{KG}^{(0)} \to \widetilde{KG}^{(0)}$ given by $g \mapsto g \cdot c$ induces a continuous, cellular, equivariant map $\phi : \widetilde{KG}^{(n)} \to \widetilde{KG}^{(n)}$ satisfying the conditions of the Σ^n-criterion. Thus $[\chi] \in \Sigma^n(G)$. $\square$

In an Artin group of finite type, Δ^2 (and perhaps Δ) is central. Thus the following is immediate:

Proposition 2.2. *Let $A_{\mathcal{G}}$ be an Artin group of finite type, and let $\chi : A_{\mathcal{G}} \to \mathbb{R}$. If $\chi(\Delta) \neq 0$, then $[\chi] \in \Sigma^n(A_{\mathcal{G}})$ for all n.*

For irreducible Artin groups of finite type, the character sphere $S(A_{\mathcal{G}})$ is a zero sphere in all but three exceptional cases, as can be seen by examining the list of irreducible Artin groups of finite type. Excluding those cases for the moment, every map $A_{\mathcal{G}} \to \mathbb{R}$ is the same as the map sending each generator to $1 \in \mathbb{Z}$ (up to scalar multiplication). In these cases, Proposition 2.2 actually shows that $\Sigma^n(A_{\mathcal{G}}) = S(A_{\mathcal{G}})$ for all n. The three exceptional cases are:

Case 1. $A_{\mathcal{G}}$ is of type $I_2(2n)$, that is, $A_{\mathcal{G}} = \langle a, b \mid (ab)^n = (ba)^n \rangle$. In this case, $A_{\mathcal{G}}^{ab} \cong \mathbb{Z} \times \mathbb{Z}$, so $S(A_{\mathcal{G}}) \cong S^1$. It then follows from [8] that the Σ-invariants are

$$\Sigma^1(A_{\mathcal{G}}) = \Sigma^n(A_{\mathcal{G}}) = S(A_{\mathcal{G}}) - \bigl\{(1,-1), (-1,1)\bigr\},$$

where $(1,-1) = \{\chi \mid \exists c > 0 \text{ such that } c\chi(a) = 1 \text{ and } c\chi(b) = -1\}$.

Case 2. $A_{\mathcal{G}}$ is of type F_4, that is,

$$A_{\mathcal{G}} = \bigl\langle a, b, c, d \mid [a,c] = [a,d] = [b,d] = 1,\ aba = bab,\ bcbc = cbcb,\ cdc = dcd \bigr\rangle.$$

Then $S(A_{\mathcal{G}}) \cong S^1$ and it follows from Theorem A′ of [16] that $\Sigma^1(A_{\mathcal{G}}) = S(A_{\mathcal{G}})$. By Proposition 2.2, $\Sigma^n(A_{\mathcal{G}}) \supset S^1 - \{(1,-1), (-1,1)\}$ for all $n > 1$.

Case 3. $A_{\mathcal{G}}$ is of type B_n where $n > 2$, that is, $A_{\mathcal{G}}$ is generated by elements $\{a_1, \ldots, a_n\}$ such that $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ for $1 \leqslant i \leqslant n-2$; $(a_{n-1}a_n)^2 = (a_n a_{n-1})^2$; and all other pairs of generators commute. A straightforward computation shows that $S(A_{\mathcal{G}}) \cong S^1$ and by [16], $\Sigma^1(A_{\mathcal{G}}) = S(A_{\mathcal{G}})$. Again by Proposition 2.2, $\Sigma^n(A_{\mathcal{G}}) \supset S^1 - \{(1,-1), (-1,1)\}$ for all $n > 1$. If $A_{\mathcal{G}}$ is of type B_3, then it is known that the commutator subgroup is *not* finitely related ((9.2) of [25]); hence $\Sigma^2(A_{\mathcal{G}}) = \Sigma^n(A_{\mathcal{G}}) = S^1 - \{(1,-1), (-1,1)\}$.

This case-by-case analysis shows that in the irreducible cases, the Σ-invariants are dense in their character spheres, with at most one pair of points removed. In general, if $A_{\mathcal{G}}$ is of finite type, one can establish that $\Sigma^n(A_{\mathcal{G}})$ is dense in $S(A_{\mathcal{G}})$ for all n by decomposing $A_{\mathcal{G}}$ into the direct product of irreducible Artin groups, and then applying the following result.

Theorem 2.3 [19,12,4]. *If G and H are groups of type $\mathcal{F}_n$, then*

$$\Sigma^n(G \times H)^c \subset \bigcup_{p+q=n} \bigl(\Sigma^p(G)^c + \Sigma^q(H)^c\bigr).$$

Our arguments will combine these observations on the Σ-invariants of Artin groups of finite type with the action of $A_{\mathcal{G}}$ on its modified Deligne complex. In particular, we will use

Theorem 2.4 [21,22]. *Let G act on an $(n-1)$-connected (respectively $(n-1)$-acyclic) complex $\mathcal{X}$ with finite quotient, and let $\chi : G \to \mathbb{R}$ be a character such that $\chi|_{G_\sigma} \neq 0$ for all cells $\sigma \in \mathcal{X}$ with $|\sigma| \leqslant n$.*

If $[\chi|_{G_\sigma}] \in \Sigma^{n-|\sigma|}(G_\sigma)$ (respectively $[\chi|_{G_\sigma}] \in \Sigma^{n-|\sigma|}(G_\sigma, \mathbb{Z})$) for all cells $\sigma \in \mathcal{X}$ with $|\sigma| < n$, then $[\chi] \in \Sigma^n(G)$ (respectively $[\chi] \in \Sigma^n(G, \mathbb{Z})$).

3. Σ^1-estimates using actions on $\mathbb{R}$-trees

Ken Brown discovered an alternate definition of the first Σ-invariant in terms of actions on $\mathbb{R}$-trees [8]. If G admits an abelian action on an $\mathbb{R}$-tree $\mathcal{T}$, then the action will fix an end e of $\mathcal{T}$. For any $x \in \mathcal{T}$ there is a unique ray $(e, x]$ representing e. If we pick a base point $b \in \mathcal{T}$, we can define a homomorphism $\chi_{\mathcal{T}} : G \to \mathbb{R}$ by $\chi_{\mathcal{T}}(g) = d(gb, x) - d(x, b)$ where $x \in \mathcal{T}$ is the unique vertex such that $(e, x] = (e, gb] \cap (e, b]$.

Theorem 3.1 [8]. *A non-zero homomorphism $\chi : G \to \mathbb{R}$ belongs to $\Sigma^1(G)$ if and only if every fixed-end G-$\mathbb{R}$-tree $\mathcal{T}$ with associated homomorphism $\chi_{\mathcal{T}} = \chi$ has a G-invariant line.*

The second author has used this result to characterize the Σ^1-invariants of graph products of groups [20]. We will use a somewhat similar argument to partially describe the Σ^1-invariants of Artin groups.

In this section let $\chi : A_{\mathcal{G}} \to \mathbb{R}$ be a character of an Artin group $A_{\mathcal{G}}$.

Let $\mathcal{L}_{\mathcal{F}} = \mathcal{L}_{\mathcal{F}}(\chi)$ be the full subgraph of $\mathcal{G}$ generated by all vertices $v \in V(\mathcal{G})$ where $\chi(v) \neq 0$. Define the *living subgraph* $\mathcal{L} = \mathcal{L}(\chi) \subseteq \mathcal{L}_{\mathcal{F}}$ to be the subgraph obtained from $\mathcal{L}_{\mathcal{F}}$ by removing those edges $\{v, w\}$ which are labeled by even integers greater than 2 and where $\chi(v) = -\chi(w) \neq 0$.

Artin groups based on edges are one-relator groups whose Σ-invariants have been completely determined [8,6]. Thus $\mathcal{L}$ can be described as the subgraph of $\mathcal{G}$ made up of all edges $e = \{v, w\}$ where $\chi(v) \neq 0 \neq \chi(w)$ and $[\chi|_{A_e}] \in \Sigma^1(A_e)$. (See Section 2, in particular Case 1, or see [16].)

Theorem 3.2 [16]. *If $\mathcal{L}(\chi)$ is a connected and dominating subgraph of $\mathcal{G}$, then $[\chi] \in \Sigma^1(A_{\mathcal{G}})$.*

A subgraph $\mathcal{D}$ of $\mathcal{G}$ is said to be *dominating* if each vertex $v \in \mathcal{G} - \mathcal{D}$ is adjacent to some vertex in $\mathcal{D}$.

Proof. Let $v \in \mathcal{G} - \mathcal{L}$ be a vertex outside $\mathcal{L} = \mathcal{L}(\chi)$. By hypothesis, $\mathcal{L}$ is dominating and thus there is a vertex $w \in \mathcal{L}$ and an edge $e = \{v, w\}$. Recall (Section 2, Case 1) that the only characters $\chi \in S(A_e) - \Sigma^1(A_e)$ are those where $\chi(v) = -\chi(w) \neq 0$; since $\chi(v) = 0$ and $\chi(w) \neq 0$, we must have $[\chi|_{A_e}] \in \Sigma^1(A_e)$. Since $\mathcal{L}$ is connected and dominating, for each $v \in \mathcal{G} - \mathcal{L}$ we can add a single edge joining v to $\mathcal{L}$; the union of $\mathcal{L}$ with these additional edges to vertices in $\mathcal{G} - \mathcal{L}$ we denote $\mathcal{L}^*$.

Let $\mathcal{T}$ be a fixed-end $A_{\mathcal{G}}$-$\mathbb{R}$-tree and let e and f be two edges in $\mathcal{L}^*$ intersecting in a common vertex $v \in \mathcal{L}$. By the construction of $\mathcal{L}^*$, we have $v \in \mathcal{L}$. Since $e, f \subset \mathcal{L}^*$, by Theorem 3.1 there is an A_e-invariant line $\ell_e \subset \mathcal{T}$ and an A_f-invariant line $\ell_f \subset \mathcal{T}$. Since $A_e \cap A_f = \langle v \rangle$, ℓ_e and ℓ_f are both $\langle v \rangle$-invariant; because $\chi(v) \neq 0$, $v \in A_{\mathcal{G}}$ is a hyperbolic element and this implies $\ell_e = \ell_f$. Consequently, the subgroup $\langle A_e, A_f \rangle$ of $A_{\mathcal{G}}$ generated by A_e and A_f has an invariant line $\ell_{\langle e,f \rangle}$ in $\mathcal{T}$. Since $\mathcal{L}^*$ is connected, continuing in this manner shows that there is a line ℓ^* in $\mathcal{T}$ which is invariant under the subgroup generated by all 'edge groups' A_e where $e \subseteq \mathcal{L}^*$. By the construction of $\mathcal{L}^*$, this subgroup contains all of the generators of our given Artin group. Thus ℓ^* is actually an $A_{\mathcal{G}}$-invariant line. □

Corollary 1 [16]. *If $\mathcal{G}$ is connected, then $\Sigma^1(A_{\mathcal{G}})$ is dense in $S(A_{\mathcal{G}})$.*

Proof. If $[\chi] \notin \Sigma^1(A_{\mathcal{G}})$, then either $\mathcal{L}(\chi)$ is not connected or not dominating by Theorem 3.2. Thus $[\chi]$ is either contained in a hyperplane of the form

$$\big\{[\chi] \in S(A_{\mathcal{G}}) \mid \chi(v) = 0 \text{ for some } v \in V(\mathcal{G})\big\},$$

or of the form

$$\big\{[\chi] \in S(A_{\mathcal{G}}) \mid \chi(v) = -\chi(w) \text{ for some } \{v, w\} \in E(\mathcal{G}) \text{ with even label } > 2\big\}. \qquad \square$$

We can apply the following theorem to establish a partial converse of Theorem 3.2.

Theorem 3.3 (Corollary 5.3 in [7]). *Let $G = A *_B C$ be a non-trivial free product with amalgamation. If $[\chi] \in \Sigma^1(G)$ then $\chi|_B$ is non-zero. In particular, if $G = A * B$, then $\Sigma^1(G)$ is empty.*

Proposition 3.4. *If $[\chi] \in \Sigma^1(A_{\mathcal{G}})$, then $\mathcal{L}_{\mathcal{F}}(\chi)$ is connected and dominating.*

Proof. Assume $\mathcal{L}_{\mathcal{F}} = \mathcal{L}_{\mathcal{F}}(\chi)$ is not connected. The quotient of $A_{\mathcal{G}}$ modulo the normal subgroup generated by all vertices in $\mathcal{G}$—$\mathcal{L}_{\mathcal{F}}$ is an Artin group based on $\mathcal{L}_{\mathcal{F}}$. The map χ factors as $\chi_{\mathcal{F}} \circ \pi$, where $\chi_{\mathcal{F}} : A_{\mathcal{L}_{\mathcal{F}}} \to \mathbb{R}$, and $\pi : A_{\mathcal{G}} \to A_{\mathcal{L}_{\mathcal{F}}}$. By the assumption that $\mathcal{L}_{\mathcal{F}}$ is not connected, $A_{\mathcal{L}_{\mathcal{F}}}$ splits as a free product, so $[\chi_{\mathcal{F}}] \notin \Sigma^1(A_{\mathcal{L}_{\mathcal{F}}})$ by Theorem 3.3. Since π is a surjection, $[\chi] \notin \Sigma^1(A_{\mathcal{G}})$ (Section I.4.1 of [7]).

Assume $\mathcal{L}_{\mathcal{F}}$ is not dominating. Let v be a vertex with $\chi(v) = 0$ which is not adjacent to $\mathcal{L}_{\mathcal{F}}$. Let $\mathcal{A}$ be the full subgraph spanned by $\mathcal{G} - \{v\}$, let $\mathcal{C}$ be the full subgraph spanned by v and its neighbors, and let $\mathcal{B} = \mathcal{A} \cap \mathcal{C}$. Then $A_{\mathcal{G}} = A_{\mathcal{A}} *_{A_{\mathcal{B}}} A_{\mathcal{C}}$ with $\chi(A_{\mathcal{B}}) = 0$, so again $[\chi] \notin \Sigma^1(A_{\mathcal{G}})$ by Theorem 3.3. $\square$

4. Proofs of the main theorems

The proofs of the main theorems combine the action of $A_{\mathcal{G}}$ on its Deligne complex, Theorem 2.4, and our comments on the Σ-invariants of Artin groups of finite type in Section 2.

Proof of Theorem A. $A_{\mathcal{G}}$ acts on its associated Deligne complex $\mathcal{DS}^f$ (see Section 1) with a copy of $\mathcal{S}^f$ as fundamental domain, and with stabilizers of the cells of $\mathcal{DS}^f$ equal to the conjugates of the Artin groups of finite type. We can not immediately apply Theorem 2.4, however, because the stabilizers of the cone points $\{a\emptyset \mid a \in A_{\mathcal{G}}\}$ are all trivial. However, we can restrict our action to the subcomplex $C\mathcal{S}^f$ consisting of cells with nontrivial stabilizers. This has the effect of removing the cone points; the complex is still 1-connected (respectively 1-acyclic) by van Kampen's Theorem (respectively by a Mayer–Vietoris argument), using the fact that $\widehat{\mathcal{G}}$ is 1-connected (respectively 1-acyclic) (see the proof of Theorem 3.2 of [17] for details).

Let $\tau \sim \{aA_{\mathcal{C}_1} \subset aA_{\mathcal{C}_2} \subset \cdots \subset aA_{\mathcal{C}_k}\}$ be a cell in $C\mathcal{S}^f$. Then $\mathrm{Stab}(\tau)$ is a conjugate of $A_{\mathcal{C}_1}$; hence by hypothesis and Proposition 2.2, for all n, $[\chi|_{\mathrm{Stab}(\tau)}] \in \Sigma^n(\mathrm{Stab}(\tau))$. Applying Theorem 2.4, $[\chi] \in \Sigma^2(A_{\mathcal{G}})$ (respectively $[\chi] \in \Sigma^2(A_{\mathcal{G}}, \mathbb{Z})$). $\square$

Proof of Corollary A.1. The kernel of $\chi : G \to \mathbb{Z}$ is $\mathcal{F}_2$ (respectively FP_2) if and only if both $[\chi]$ and $[-\chi]$ are elements of $\Sigma^2(G)$ [respectively $\Sigma^2(G, \mathbb{Z})$] by [6] or [23]. Also, if $\alpha \in \mathrm{Aut}(G)$, then $[\chi] \in \Sigma^2$ if and only if $[\chi \circ \alpha] \in \Sigma^2$.

Due to the symmetry of the defining relations of $A_{\mathcal{G}}$, the map which sends each generator v to its inverse extents to an automorphism. Since $\chi \circ \alpha = -\chi$ for all $\chi : A_{\mathcal{G}} \to \mathbb{Z}$, the result follows from Theorem A and the previous remarks. $\square$

Proof of Corollary A.2. By Theorem A, if $[\chi] \notin \Sigma^2(A_{\mathcal{G}})$, then $\chi(\Delta_\sigma) = 0$ for some σ. But the condition $\chi(\Delta_\sigma) = 0$ corresponds to a linear equation in the generators. (For example, if σ is a single edge labeled 4, then $\Delta_\sigma = abab$ and $\chi(\Delta_\sigma) = 0$ implies $\chi(a) = -\chi(b)$.) So, $\Sigma^2(A_{\mathcal{G}})^c$ must sit inside the intersection of $S(A_{\mathcal{G}})$ with the union of the hyperplanes in $\mathrm{Hom}(\mathrm{A}_{\mathcal{G}}, \mathbb{R})$ described by $\{\chi(\Delta_\sigma) = 0 \mid \sigma \text{ is a simplex in } \widehat{\mathcal{G}}\}$.

The proof in the homological case is similar. $\square$

Proof of Theorem B. By [11], if $A_{\mathcal{G}}$ is of FC type, the Deligne complex $\mathcal{DS}^f$ is contractible. The result now follows by the same arguments as in the proof of Theorem A. $\square$

Proof of Corollaries B.1 and B.2. The proofs of Corollaries A.1 and A.2 hold, mutatis mutandis. $\square$

5. The Σ-invariants for Artin groups based on trees

For Artin groups $A_{\mathcal{T}}$ with defining graph $\mathcal{T}$ a tree, the first author determined $\Sigma^1(A_{\mathcal{T}})$ in a more general context in [16]. Similar arguments as in the proofs of our main results can then be used to give a complete description of the higher invariants. However, in the following we will give a new, complete, and unified proof much in the spirit of the techniques in [18].

Theorem 5.1. *Let $A_{\mathcal{T}}$ be an Artin group based on a tree $\mathcal{T}$. Then*:

(i) $\Sigma^1(A_{\mathcal{T}}) = \{[\chi] \in S(A_{\mathcal{T}}) \mid \mathcal{L}(\chi) \subseteq \mathcal{T}$ is connected and dominating$\}$;

(ii) $\Sigma^1(A_{\mathcal{T}}) = \Sigma^n(A_{\mathcal{T}}) = \Sigma^n(A_{\mathcal{T}}, \mathbb{Z})$ *for all* $n \geqslant 2$.

Proof. Let Y be a maximal tree in the nerve of the covering of $\mathcal{T}$ by the set of all edges. Recall that the vertices of the nerve are the edges of $\mathcal{T}$ and that two edges of $\mathcal{T}$ form an edge in the nerve if they intersect in a common vertex in $\mathcal{T}$. Thus the nerve is a finite graph with Y as a maximal subtree.

To each vertex $V \in V(Y)$ we associate the group $G(V) = A_e \leqslant A_{\mathcal{T}}$ if V is represented by the edge $e \in E(\mathcal{T})$, and to each edge $E \in E(Y)$, represented by the vertex $v \in V(\mathcal{T})$, we associate the infinite cyclic group $G(E) = \langle v \rangle \leqslant A_{\mathcal{T}}$. Since $\mathcal{T}$ is a tree, $A_{\mathcal{T}}$ is the fundamental group $\pi(G(-), Y)$ of this finite tree of groups.

Suppose that $[\chi] \in \Sigma^1(A_{\mathcal{T}})$. By Proposition 3.4, $\mathcal{L}_{\mathcal{F}}(\chi)$ is connected and dominating and hence $\chi(v) \neq 0$ for each vertex of degree greater than one. Equivalently, $\chi|_{G(E)}$ is non-zero for each edge $E \in E(Y)$, and hence $[\chi|_{G(E)}] \in \Sigma^n(G(E))$ for all n because $G(E)$ is infinite cyclic. Using a result on group actions on trees (Theorem II.5.1 in [7,24], or Theorem 9.1 in [17]), we find that $[\chi|_{G(V)}] \in \Sigma^1(G(V))$ for each vertex $V \in V(Y)$.

By construction, this is equivalent to saying that $[\chi|_{A_e}] \in \Sigma^1(A_e)$ for each edge $e \in E(\mathcal{T})$. Consequently, $\mathcal{L}(\chi) = \mathcal{L}_{\mathcal{F}}(\chi)$ is connected and dominating. Conversely, if $\mathcal{L}(\chi)$ is connected and dominating, $[\chi] \in \Sigma^1(A_{\mathcal{T}})$ by Theorem 3.2.

Let $n \geqslant 2$. Since A_e is a one-relator group for all edges $e \in E(\mathcal{T})$, $\Sigma^1(A_e) = \Sigma^n(A_e) = \Sigma^n(A_e, \mathbb{Z})$ (see [6]). Thus $[\chi|_{G(V)}] \in \Sigma^n(G(V))$ for each vertex $V \in V(Y)$. As shown above, $[\chi|_{G(E)}] \in \Sigma^n(G(E))$ for each edge $E \in E(Y)$. Thus applying Theorem 2.4 gives the desired result. $\square$

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